FIITJEE

KKP / MYP CENTRE

OLYMPIAD FOUNDATION (PRMO)

KEY

1.	14	**2.**	10	**3.**	33	**4.**	9	**5.**	54
6.	60	**7.**	10	**8.**	3	**9.**	6	**10.**	6
11.	30	**12.**	2	**13.**	38	**14.**	5	**15.**	0
16.	18	**17.**	24	**18.**	69	**19.**	59	**20.**	49
21.	18	**22.**	4	**23.**	38	**24.**	0	**25.**	10
26.	14	**27.**	51	**28.**	5	**29.**	13	**30.**	3

Solutions

10.05.2020

1) Let the two numbers be x and $x+2$

Given that $4x-10 \le 2(x+2)$

$\Rightarrow 2x \le 14$

$\Rightarrow x \le 7$

Since x is even, the largest such $x = 6$.

$\therefore$ the numbers are 6 and 8.

$\therefore$ required $= 6+8=14$

2) $x_{n+1}+x_n = 100 \longrightarrow (1)$

$\therefore x_{n+2}+x_{n+1} = 100 \longrightarrow (2)$

$\therefore (1)-(2) \Rightarrow x_{n+2} = x_n$

$\therefore x_{50} = x_{48} = x_{46} = \cdots = x_{10} = 10$

3) We have, $x = 100-3y$.

Since x is a natural number, $x \ge 1$

$\Rightarrow 100-3y \ge 1$

$\Rightarrow y \le 33$

$\therefore$ for each $y \in \{1, 2, \ldots, 33\}$; we get exactly one natural number 'x'.

$\therefore$ Number of ordered pairs $= 33$

4) LHS = RHS = 0 if $x = 0, 1, 2, 3, 4$

LHS = RHS = 1 if $x = 7, 8, 9$

LHS = RHS = 2 if $x = 14$

And for any other positive integer x, $\left[\frac{x}{5}\right] \neq \left[\frac{x}{7}\right]$

$\therefore$ required number $= 9$

5) Let $a+1 = b+2 = c+3 = d+4 = a+b+c+d-8 = t$

$\therefore$ $a = t-1$, $b = t-2$, $c = t-3$, $d = t-4$

$\therefore$ $a+b+c+d-8 = t$

$\Rightarrow$ $4t - 10 - 8 = t$ $\Rightarrow$ $t = 6$

$\therefore$ $a = 5$, $b = 4$, $c = 3$, $d = 2$, $e = 1$

$\therefore$ $a^2 + b^2 + c^2 + d^2 = 54$

6) Given, $x + y = 12 \longrightarrow (1)$

$y + z = 15 \longrightarrow (2)$

$z + x = 11 \longrightarrow (3)$

$\therefore$ $(1) + (2) + (3) \Rightarrow x + y + z = 19 \longrightarrow (4)$

$\therefore$ $(4) - (1) \Rightarrow z = 7$

$(4) - (2) \Rightarrow x = 4$

$(4) - (3) \Rightarrow y = 8$

$\therefore$ $2x + 3y + 4z = 60$.

7) $a^b = 1$ iff either i) $a \neq 0, b = 0$ (or)

ii) $a = 1$, b is any real

$\therefore (x^2-5x+7)^{x^2-8x+15}=1$ iff

either i) $x^2-8x+15=0$ and $x^2-5x+7 \neq 0$

$\Rightarrow x = 3$ (or) 5 and 'x is any real'

$\therefore x=3$ and 5 are two of the solutions

or ii) $x^2-5x+7=1$

$\Rightarrow x^2-5x+6=0 \Rightarrow x=2$ (or) 3

$\therefore$ the solution set is $\{2,3,5\}$

$\therefore$ sum of all solutions $= 2+3+5=10$

8) $a^2+4b^2=5ab \Rightarrow (a+2b)^2 = 9ab$

$\Rightarrow (a-2b)^2 = ab$

$\therefore \dfrac{(a+2b)^2}{(a-2b)^2} = 9$

$\Rightarrow \left|\dfrac{a+2b}{a-2b}\right| = 3$

9) Let $n-3=t \Rightarrow n=t+3$

$\therefore \dfrac{(n+1)(n+2)}{n-3} = \dfrac{(t+4)(t+5)}{t}$

$= t+9+\dfrac{20}{t}$

$= n+6+\dfrac{20}{n-3}$

This is a natural number if $n-3 = 1, 2, 4, 5, 10, 20$

$\therefore n = 4, 5, 7, 8, 13, 23$

$\therefore n(A) = 6$

10) n^n is a perfect square if either n is even (or) n is a perfect square.

$\therefore n = 2, 4, 6, \ldots, 2020$ (or)

$1^2, 3^2, 5^2, \ldots, 43^2.$

$\therefore n(A) = 1010 + 22 = 1032.$

$\therefore$ Sum of the digits of $n(A) = 6.$

11) We have, $2|70-25| = 70+a \Rightarrow a = 20$

$\therefore 2|x-25| = x+20.$

$\therefore$ if $x \le 25$, then $2(25-x) = x+20 \Rightarrow x = 10$

$\therefore a = 20$ and $b = 10$

$\therefore a+b = 30$

12) Case(i): If $x \ge 0, y \ge 0$,

then $2x+y = 8$ and $x = 14$

$\therefore y = -20 < 0$ not possible.

Case(ii): if $x \ge 0, y \le 0$,

then $2x+y = 8$ and $x-2y = 14$

$\therefore x = 6, y = -4$

Case(iii): if $x \le 0, y \ge 0$

then, $y = 8$

$\therefore x = 14 > 0$ not possible.

Case (iv): if $x \le 0$, $y \le 0$

then $y = 8 > 0$ not possible.

$\therefore$ the solution is $x = 6$ and $y = -4$

$\therefore x + y = 2$

13) $[x] = \frac{x^2+3}{5}$

Since LHS is an integer, RHS must be integer.

$\therefore x^2 = 2, 7, 12, 17$ only satisfy

$\therefore$ required $= 2+7+12+17$

$= 38$.

14) Both a and b should be of the form $5K$ (or) $5K-1$ (or) $5K-2$ (or) $5K-3$ (or) $5K-4$, where $K = 1, 2, \dots, 20$.

$\therefore$ we need five colours.

15) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1$

$$\Rightarrow \frac{a(a+b+c)}{b+c} + \frac{b(a+b+c)}{c+a} + \frac{c(a+b+c)}{a+b} = a+b+c$$

$$\Rightarrow \frac{a^2}{b+c} + a + \frac{b^2}{c+a} + b + \frac{c^2}{a+b} + c = a+b+c$$

$$\Rightarrow \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} = 0$$

16) $|a-1|<2 \Rightarrow -1<a<3$

$|b-3|<4 \Rightarrow -1<b<7$

$|c-5|<6 \Rightarrow -1<c<11$

$\therefore -3 < a+b+c < 21$

$\therefore x=-3,\ y=21$

$\therefore x+y=18$

17) $|2-a-|6-|a-20||| = |2-a-|6-(a-20)||$

$= |2-a-|26-a||$

$= |2-a-(26-a)|$

$= |2-26| = 24$

18) if $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

$\therefore \frac{33}{100} < \frac{34}{103} < \frac{1}{3}$

$\therefore m=34,\ n=103$

$\therefore n-m=69$

19) Smallest $x = L.C.M(3,4,5)-1$

$= 60-1 = 59$

20) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right) \text{----} \left(1-\frac{1}{99^2}\right)$

$= (1+1/2)(1-1/2)(1+1/3)(1-1/3) \text{.....} \left(1+\frac{1}{99}\right)(1-1/99)$

$= (1+1/2)(1+1/3)\cdots(1+1/99)\cdot(1-1/2)(1-1/3)\cdots(1-1/99)$

$= \left(\frac{3\cdot4\cdot5\cdots\cdots100}{2\cdot3\cdot4\cdots\cdots99}\right)\left(\frac{1\cdot2\cdot3\cdots98}{2\cdot3\cdot4\cdots\cdots99}\right)$

$= (50)\left(\frac{1}{99}\right) = \frac{50}{99}$

$\therefore p = 50,\ q = 99$

$\therefore q - p = 49$

21) $a - b = \pm 2 \longrightarrow ①$

$b - c = \pm 3 \longrightarrow ②$

$c - d = \pm 4 \longrightarrow ③$

Adding ①, ② and ③, we get

$a - d = \pm 9,\ \pm 1,\ \pm 3,\ \pm 5$

$\therefore |a-d| = 9, 1, 3, 5$

$\therefore$ required $= 9+1+3+5 = 18$

22) Median of $1, 2, 3, 4$ is $\frac{2+3}{2} = 5/2$

$\therefore$ minimum $= \left|\frac{5}{2}-1\right| + \left|\frac{5}{2}-2\right| + \left|\frac{5}{2}-3\right| + \left|\frac{5}{2}-4\right|$

$= \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + \frac{3}{2} = 4$

23) Case(i): if $x \geq y$ then $3x + 2y = 16$

$4x + 3y = 5$

$\therefore x = 38,\ y = -49$

But $|x| \ngtr |y|$

$\therefore$ we can't take this solution

Case(ii): if $x < y$ then $x + 4y = 16$

$2x + 5y = 5$

$\therefore \quad x = -20, \quad y = 9$

$\therefore |x - 2y| = |-20 - 18| = 38$

24) $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2021}$

$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2021}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2020}\right)$

$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2021}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1010}\right)$

$= \frac{1}{1011} + \frac{1}{1012} + \cdots + \frac{1}{1021}$

$\Rightarrow S = P.$

$\therefore \quad (S - P)^{2022} = 0$

25) The smallest natural number 'n' that satisfies given conditions is $n = 5$.

Now, L.C.M.$(5, 6, 7) = 210$.

$\therefore$ required numbers are $5, 5 + 210, 5 + 2(210), \ldots$

$\therefore$ number of such numbers less than (or) equal to 2020 is 10.

26) Let $f(x) = |x-2|+|x-4|+|x-6|+|x-8|+|x-10|+|x-12|$.

The minimum value of $f(x)$ exists at the median of $2, 4, 6, 8, 10, 12$. And the minimum value is 18. As there are two middle values 6 and 8, $f(x) = 18, \ \forall x \in [6, 8]$

$$\therefore a = 6, \ b = 8$$

$$\therefore a + b = 14.$$

27) $x = ||\ldots|||99-1|-2|-3| \ldots\ldots -98|-99|$

$= ||\ldots|||99-(1+2+\cdots+13)|-14|-15| \ldots\ldots -98|-99|$

$= ||\ldots\ldots|||8-14|-15|-16| \ldots\ldots -98|-99|$

$= ||\ldots||6-15|-16| \ldots\ldots -98|-99|$

$= 99-(98-(97-(96-(\ldots\ldots(16-(15-6))\ldots\ldots))))$

$= (99-98)+(97-96)+\ldots\ldots+(17-16)+9$

$= 42+9$

$= 51$

28) Given, $[x]-[y] = \frac{2}{3}+\{y\} \longrightarrow (1)$ $(\because y = [y]+\{y\})$

$2[y]-[z] = \frac{2}{3}+\{z\} \longrightarrow (2)$

$3[z]-[x] = \frac{2}{3}+\{x\} \longrightarrow (3)$

Since LHS of (1), (2) and (3) are all integers,

$\{y\} = \{z\} = \{x\} = 1/3$

$\therefore [x]-[y]=1$

$2[y]-[z]=1$

$3[z]-[x]=1$

Solving these equations, we get $[x]=2$, $[y]=[z]=1$

$\therefore x = 7/3$, $y=4/3$, $z=4/3$

$\therefore x+y+z=5$

29) $\frac{p}{q} = \frac{2^2 \cdot 3^2 \cdot 4^2 \cdots\cdots 2020^2}{(1\cdot3)(2\cdot4)(3\cdot5)\cdots\cdots(2019)(2021)}$

$= \frac{2(2020)}{2021} = \frac{4040}{2021}$

$\therefore p = 4040$, $q=2021$

$\therefore p+q = 6061$

$\therefore$ Sum of the digits in $p+q$ is 13

30) Let $n=p^2$ and $n+200=q^2$; where $p,q \in \mathbb{N}$

$\therefore q^2-p^2=200$

$\Rightarrow (q+p)(q-p)=200$

$\Rightarrow (q+p)(q-p)=2^3 5^2$

Since, $q+p > q-p$, we have the following possibilities :

i) $q+p=200$ and $q-p=1$ $\Longrightarrow$ not possible

ii) $q+p=100$ and $q-p=2$ $\Rightarrow (q,p)=(51,49)$

iii) $q+p=50$ and $q-p=4$ $\Rightarrow (q,p)=(27,23)$

iv) $q+p=25$ and $q-p=8 \Rightarrow$ not possible

v) $q+p=40$ and $q-p=5 \Rightarrow$ not possible

vi) $q+p=20$ and $q-p=10 \Rightarrow (q,p)=(15,5)$

$\therefore\ p = 49, 23, 5$

$\therefore\ n = 2401, 529, 25$