QUADRATIC EQUATIONS

Quadratic Expression

A polynomial of the form ax^2+bx+c, where a,b,c are real or complex numbers and $a\neq 0$, is called a quadratic expression in variable $'x'$.

A complex number $'k'$ is said to be a zero of quadratic expression ax^2+bx+c if $ak^2+bk+c=0$

Quadratic equations in one variable

Any equation of the form $ax^2+bx+c=0$ where a,b,c are real or complex numbers and $a\neq 0$ is called a quadratic equation in the variable $'x'$. The numbers a,b,c are called coefficients of the quadratic equation.

C1 Solving Q.E and Relation between the Roots

Formula Method

To find roots of quadratic equation

Let $ax^2+bx+c=0$ is given quadratic equation. Then

$4a^2x^2+4abx+4ac=0$ (by multiplying with $4a$ both sides)

$(2ax+b)^2=b^2-4ac \quad \Rightarrow 2ax+b=\pm\sqrt{b^2-4ac}$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$\therefore$ The roots of the quadratic equation

$ax^2+bx+c=0$ are $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Let they are α,β and assume $\alpha=\dfrac{-b-\sqrt{b^2-4ac}}{2a}, \beta=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$

Then $\alpha+\beta=-\dfrac{b}{a}$; $\alpha\beta=\dfrac{c}{a}$

$$\therefore ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left(x^2-(\alpha+\beta)x+\alpha\beta\right) = a(x-\alpha)(x-\beta)$$

$\therefore$ The quadratic equation having roots α,β is $(x-\alpha)(x-\beta)=0$

or $x^2-(\alpha+\beta)x+\alpha\beta=0$ or $x^2-Sx+P=0$

Where $S=$ sum of roots, $P=$ product of roots.

Note1 An equation of the form $ax^2+bx+c=0$ has more than two roots then it is an identity i.e., $a=0,b=0,c=0$ and it is true for all values of x

Note2 If $\alpha+i\beta$ is a root of real quadratic equation then $\alpha-i\beta$ is also a root.

Note3 If $\alpha+\sqrt{\beta}$ is a root of rational quadratic equation then $\alpha-\sqrt{\beta}$ is also a root where $\beta\neq k^2$.

Practice Problems

1. Find the roots of the equation

   (a) $x^2+2x-5=0$ (b) $x^2+2x-24=0$ (c) $x^2+4x+13=0$

2. Find the quadratic equation with leading coefficient one whose roots are

   (a) $2+\sqrt{3}$ and $2-\sqrt{3}$ (b) $1+2i$ and $1-2i$

3. Find a quadratic equations with real coefficients for which one of the root is $2+3i$

4. Find a quadratic equation whose roots are $2+3i$ and 2

5. Solving the following equations

   i) $x^4-5x^2+6=0$ ii) $x^{2/3}+x^{1/3}-2=0$ iii) $3^{1+x}+3^{1-x}=10$

   iv) $4^{x-1}-3.2^{x-1}+2=0$ v) $2\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+5=0,\ x\neq 0$

   vi) $\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0,\ x\neq 0$ vii) $\left(x+\frac{1}{x}\right)^2-\frac{3}{2}\left(x-\frac{1}{x}\right)=4$

   viii) $(x-1)(x-3)(x-5)(x-7)=9$ (Hint: $M-1: x^2-8x=t$; $M-2: x-4=t$)

   ix) $2x^4+x^3-11x^2+x+2=0$ x) $\sqrt{\frac{x}{x-3}}+\sqrt{\frac{x-3}{x}}=\frac{5}{2},\ x\neq 0$

6. Solve the following

   i) $\sqrt{3x+1}-\sqrt{x-1}=2$ ii) $\sqrt{2x+1}+\sqrt{3x+2}=\sqrt{5x+3}$ iii) $\sqrt{x+1}+\sqrt{2x-5}=3$

7. Solve the following

   i) $\sqrt{3x-8}<-2$ ii) $\sqrt{-x^2+6x-5}>8-2x$ iii) $\sqrt{x+2}>\sqrt{8-x^2}$

   iv) $\frac{\sqrt{6+x-x^2}}{2x+5}\geq\frac{\sqrt{6+x-x^2}}{x+4}$ vi) $\sqrt{x^2-3x-10}>8-x$

8. If α,β are the roots of $x^2+2x+4=0$ find

   (a) $\frac{1}{\alpha}+\frac{1}{\beta}$ (b) $\alpha^4+\beta^4$ (c) $\frac{\alpha+1}{\beta}+\frac{\beta+1}{\alpha}$

   (d) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$ (e) $\alpha^5+\beta^5$ (f) $(6-\alpha)(6-\beta)$

9. Let $\alpha\neq\beta$ and $\alpha^2+3=5\alpha$ and $\beta^2=5\beta-3$ then find the quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

10. If one root of $x^2-x-k=0$ is square of the other then find the value of 'k'.

11. If α and β are the roots of $ax^2+bx+c=0$ then find

    (a) $\frac{1}{a\alpha+b}+\frac{1}{a\beta+b}$ (b) $\frac{\beta}{a\alpha+b}+\frac{\alpha}{a\beta+b}$ (c) $(a\alpha+b)^{-2}+(a\beta+b)^{-2}$

12. If α and β are the roots of $ax^2+bx+c=0$ find the equation whose roots are

(a) $\frac{1}{\alpha+\beta}, \frac{1}{\alpha}+\frac{1}{\beta}$ (b) $\alpha+\frac{1}{\beta}, \beta+\frac{1}{\alpha}$ (c) $\alpha^2+\beta^2, \frac{1}{\alpha^2}+\frac{1}{\beta^2}$

13. Solve the equation $\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1$

14. If α,β are the roots the equation $x^2+px+1=0$ and γ,δ are the roots of the equation $x^2+qx+1=0$ then find the value of $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$

15. Given that α,γ be the roots of $Ax^2+4x+1=0$ and β,δ are the roots of $Bx^2-6x+1=0$, find the values of A and B such that $\alpha,\beta,\gamma,\delta$ are in H.P

Answer & Key

1. (a) $-1\pm\sqrt{6}$ (b) $4,-6$ (c) $-2+3i,-2-3i$

2. (a) $x^2-4x+1=0$ (b) $x^2-2x+5=0$

3. $x^2-4x+13=0$ 4. $x^2-(4+3i)x+(4+6i)=0$

5. i) $\pm\sqrt{2},\pm\sqrt{3}$ ii) $-8,1$ iii) $1,-1$ iv) $1,2$

v) $2,\frac{1}{2},\frac{1\pm i\sqrt{3}}{2}$ vi) $2\pm\sqrt{3},\frac{1\pm i\sqrt{3}}{2}$ vii) $-1,-\frac{1}{2},1,2$ viii) $4,4,4\pm\sqrt{10}$

ix) $\frac{1}{2},2,\frac{-3\pm\sqrt{5}}{2}$ x) $-1,4$

6. i) $x=5,1$ ii) $x=-\frac{1}{2}$ iii) $x=3$

7. i) $x\in\phi$ ii) $x\in(3,5]$ iii) $x\in\left(2,2\sqrt{2}\right]$ iv) $x\in[-2,-1]\cup\{3\}$

v) $x\in\left(\frac{74}{13},\infty\right)$

8. (a) $-\frac{1}{2}$ (b) -16 (c) $-\frac{3}{2}$ (d) $\frac{1}{4}$ (e) -32 (f) 52

9. $\frac{19}{3}$ 10. $2\pm\sqrt{5}$ 11. (a) $\frac{b}{ac}$ (b) $-\frac{2}{a}$ (c) $\frac{b^2-2ac}{a^2c^2}$

12. (a) $bcx^2+(a+b)x+ab=0$ (b) $acy^2+(a+c)by+(a+c)^2=0$

(c) $c^2x^2+\left(b^2-2ac\right)\left(c^2-a^2\right)x+\left(b^2-2ac\right)^2=0$

13. Every complex number 14. q^2-p^2

15. $A=-21, B=-16$

C2 Nature of roots

For the quadratic equation $ax^2+bx+c=0$, b^2-4ac is called the discriminant of the quadratic equation denoted by D or Δ.

$\therefore D=b^2-4ac$

If a,b,c are real then the nature of roots of equation is as follows

1. If $b^2-4ac=0$, then the roots of the equation are real and equal
2. If $b^2-4ac>0$, then the roots of the equation are real and distinct
3. If $b^2-4ac<0$, then the roots of the equation are non real conjugate complex numbers

Practice Problems

1. Find the nature of the roots of the equation

   (a) $2x^2+3x+7=0$ (b) $2x^2+x-3=0$

2. If $x^2-2(1+3m)x+7(3+2m)=0$ has equal roots then find the value of $'m'$.

3. Prove that the biquadratic equation $\left(ax^2+bx+c\right)\left(ax^2-bx-c\right)=0$ has at least two real roots.

4. If the roots of the equation $x^2-8x+a^2-6a=0$ are real and distinct then find the range of $'a'$.

5. If $a+b+c=0$, then find the nature of the roots of the equation $4ax^2+3bx+2c=0, a\neq 0$

6. Show that if p,q,r,s are real numbers and $pr=2(q+s)$ then atleast one of the equations $x^2+px+q=0$, $x^2+rx+s=0$ has real roots

7. Prove that the roots of the equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are always real and they will be equal if and only if $a=b=c$

8. If the roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are equal, prove that a,b,c are in H.P

9. If $a<b<c<d$, then prove that for every real λ, the quadratic equation $(x-a)(x-c)+\lambda(x-b)(x-d)=0$ has real roots.

10. If a and b are odd integers, then find the number of real roots of $\left[x^2\right]+a[x]+b=0$, where $[.]$ denotes 'Greatest Integer Function'.

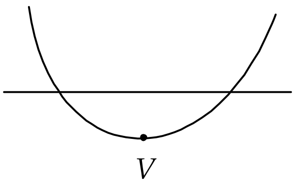
Answer & Key

1.	(a) non real complex	(b) real and distinct		2.	$2,-\frac{10}{9}$
4.	$a\in(-2,8)$	5.	Real & distinct	10.	Zero

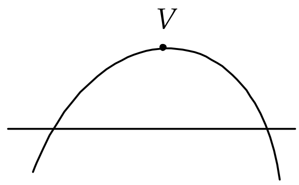
C3 Graph & Sign changes of a Quadratic Expression

The graph of $y = ax^2 + bx + c$ is as follows

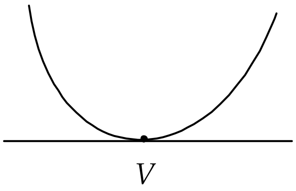
(i) $a > 0, D > 0$



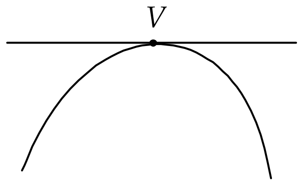
(ii) $a < 0, D > 0$



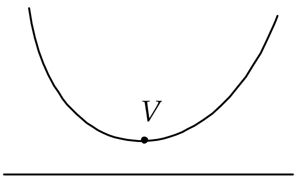
(iii) $a > 0, D = 0$



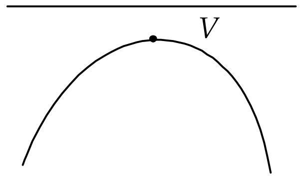
(iv) $a < 0, D = 0$



(v) $a > 0, D < 0$



(vi) $a < 0, D < 0$



The value of V in each figure is $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

Practice Problems

1. If $ax^2 + bx + c = 0$ has imaginary roots and $a + c < b$, then prove that $4a + c < 2b$.

2. If the equation $ax^2 + 2bx + c = 0$ has non real roots then prove that $a^2 + c^2 > 2b^2$.

3. If $ax^2 + bx + 6 = 0$ doesn't have distinct real roots then find the least value of $3a + b$

4. Let $f(x) = ax^2 + bx + c$ be a quadratic expression having its vertex at $(3, -2)$ and the value of $f(0) = 7$ then find $f(x)$.

5. Find the sum of the abscissa of the points where the curves $y = 3kx^2 - (5k - 8)x + 2k - 5$, $k \in \mathbb{R}$ touch the x-axis

6. If $(-2, 2020)$ is the highest point on the graph of $y = -2x^2 - 4ax + k$ then find $'k'$

7. Draw the graphs of the following

   i) $f(x) = x^2 - 5x + 6$ ii) $f(x) = 2x^2 - 12x + 18$ iii) $f(x) = 3x^2 + 3x + 1$

   iv) $f(x) = -x^2 + 3x + 4$ v) $f(x) = -x^2 - 10x - 25$ vi) $f(x) = -2x^2 - 2x - 2$

8. Discuss the sign changes of the following Quadratic Expressions, where x is real

i) x^2-5x+4 ii) x^2-x+3 iii) $3x^2+4x+4$

iv) $-5x^2+4x+2$ v) $x^2-7x+10$ vi) $15+4x-3x^2$

9. Solve the following inequalities

i) $15x^2+4x-4\le 0$ ii) $x^2-2x+1<0$ iii) $x^2-4x-21\ge 0$

10. If $f(x)=ax^2+bx+c$, $a\ne 0$, $a,b,c\in\mathbb{R}$ then write necessary and sufficient conditions for the following statements

i) $f(x)\ge 0, \forall x\in\mathbb{R}$ ii) $f(x)>0, \forall x\in\mathbb{R}$

iii) $f(x)\le 0, \forall x\in\mathbb{R}$ iv) $f(x)<0, \forall n\in\mathbb{R}$

Answer & Key

3.	-2	4.	$f(x)=x^2-6x+7$	5.	C	6.	2012

PROGRESSION

C1 Arithmetic Progression (A.P)

A sequence of terms $t_1, t_2, t_3 ...., t_n, ...$ are said to be in A.P if

$t_{k+1}-t_k=$ constant $\forall k\in\mathbb{N}$. The first term 't_1' in an A.P is usually denoted by 'a' and the constant difference between two successive terms by 'd' called common difference of the A.P. Therefore the A.P is of the form $a, a+d, a+2d, a+3d, ...$

n^{th} term of an A.P

The nth term of an A.P denoted by 't_n' is given by

$$t_n=a+(n-1)d$$

Note: Since 'a' and 'd' are constants and $t_n=a+(n-1)d$, the nth term of an A.P is linear expression in 'n'

Practice Problems

1. In an A.P if $t_n=5n+3$ then find 'd'.

2. In an A.P of $mt_m=nt_n$ then find t_{m+n}

3. If $t_m=\dfrac{1}{n}$ and $t_n=\dfrac{1}{m}$ then find t_{mn}

4. In an A.P if $\dfrac{t_4}{t_7}=\dfrac{2}{3}$ then find $\dfrac{t_6}{t_8}$

5. Find the number of identical terms in the two AP's

2, 5, 8, 11, …….. 60 terms and 3, 5, 7, 9, ….. 50 terms

6. The interior angles of a polygon are in arithmetic progression. The smallest angle is $120°$ and the common difference is $5°$. Find the number of sides of the polygon.

7. If $a_1, a_2, a_3, ..., a_n$ are non zero terms which are in A.P then prove $\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + ... + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}$

8. In an A.P if $t_5 = 17, t_7 = 15$ then find t_{22}

C2. Sum of 'n' terms of an A.P

The sum of first 'n' terms of an A.P is denoted by S_n and is given by

$$S_n = a + (a+d) + (a+2d) + ... + (a+(n-1)d)$$

$$= na + d(1+2+3+.....+n-1)$$

$$= na + d\frac{(n-1)n}{2}$$

$$= \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[\text{first term + Last term}]$$

Note: S_n is of the form $an + bn^2$ where a, b are constants

Practice Problems

1. In an A.P if $S_n = 3n^2 + 4n$, find t_{10}

2. In an A.P if $S_m = n, S_n = m$ then find S_{m+n}

3. Find the sum of all three digit natural numbers which are divisible by 7

4. Find the sum of integers from 1 to 100 which are divisible by either 2 or 5

5. In an A.P if $t_1 = 20$, $t_p = q$, $t_q = p$ find the value of $'m'$ such that sum of the first $'m'$ terms of the A.P is zero

6. Find the number of terms of an A.P series $20, 19\frac{1}{3}, 18\frac{2}{3}$, ….. of which the sum of terms is 300.

7. If the ratio of the sum of $'n'$ terms of two AP's is $\frac{7n+1}{4n+27}$, then find the ratio of their 12th terms.

8. If S_1, S_2, S_3 are the sum of first $n, 2n, 3n$ terms respectively of an A.P then show that $S_3 = 3(S_2 - S_1)$.

9. If the sum of first 8 and 19 terms of an A.P are 64 and 361 respectively, find the common difference and S_n.

10. Let S_n denote the sum of first $'n'$ terms of an A.P. If $S_{2n} = 3S_n$ then the find the ratio $\frac{S_{3n}}{S_n}$

11. Find the least value of $'n'$ for which the sum $2+5+8+....+n$ terms exceeds 1000.

12. Find the maximum value of the sum of the series $10+9\frac{1}{3}+8\frac{2}{3}+......$

13. The number of terms of an A.P is even, the sum of odd terms is 24, of even terms is 30, and the last term exceeds the first term by $\frac{21}{2}$ then find the number of terms

14. If $S_1, S_2, ...S_m$ are the sum of 'n' terms of m AP's whose first terms are 1,2,3,.....,m and the common differences are 1,3,5,......$(2m-1)$. Then find $S_1+S_2+.......+S_m$

15. The set of natural numbers is divided into sets $S_1=\{1\}$, $S_2=\{2,3\}$, $S_3=\{4,5,6\}$, $S_4=\{7,8,9,10\}$ and soon. Then find the sum of elements in the set S_{50}.

C3. Properties of an A.P

1. If $a_1, a_2, a_3, ..., a_{2n}$ are in A.P then

 (a) $a_1+a_{2n}=a_2+a_{2n-1}=.....=a_n+a_{n+1}$

 (b) $a_n=\frac{a_{n-k}+a_{n+k}}{2}, k=0,1,2,....,n$

2. If three terms a,b,c are in A.P then $2b=a+c$

3. (a) Three terms in A.P are taken as $a-d, a, a+d$

 (b) Four terms in A.P are taken as $a-3d, a-d, a+d, a+3d$

4. If $a_1, a_2, a_3, ..., a_n$ are in A.P then

 (a) $a_1+k, a_2+k, a_3+k, ..., a_n+k$ are in A.P $\forall\, k \in \mathbb{R}$

 (b) $a_1k, a_2k, a_3k, ..., a_nk$ are also in A.P $\forall\, k \in \mathbb{R}$

5. If $a_1, a_2, a_3, ..., a_n$ and $b_1, b_2, b_3, ..., b_n$ are two A.P's then

 $pa_1+qb_1, pa_2+qb_2, pa_3+qb_3, ..., pa_n+qb_n$ are also in A.P $\forall\, p,q \in \mathbb{R}$

Practice Problems

1. If a,b,c are in A.P, prove that the following are also in A.P

 (a) $b+c, c+a, a+b$

 (b) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$

 (c) $a+\frac{1}{bc}, b+\frac{1}{ca}, c+\frac{1}{ab}$

 (d) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$

 (e) $bc-a^2, ca-b^2, ab-c^2$

 (f) $a^2(b+c), b^2(c+a), c^2(a+b)$

 (g) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$

 (h) $(b+c)^2-a^2, (c+a)^2-b^2, (a+b)^2-c^2$

2. If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P, then prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are also in A.P

C4. Arithmetic Means (A.M's)

Let $'a'$ and $'b'$ are two quantities. If $x_1, x_2, ..., x_n$ are $'n'$ quantities such that $a, x_1, x_2, ..., x_n, b$ are in A.P then $x_1, x_2, ..., x_n$ are called $'n'$ arithmetic means (A.M's) between $'a'$ and $'b'$

Here $b = t_{n+2} = a + (n+1)d$

$\Rightarrow$ common difference $d = \dfrac{b-a}{n+1}$

Sum of $'n'$ A.M's

$= x_1 + x_2 + ... + x_n = \dfrac{n}{2}[x_1 + x_n]$

$= \dfrac{n}{2}[a+b]$

Practice Problems

1. Between '1' and '31' are inserted $'m'$ arithmetic means so that the ratio of the 7th and $(m-1)$th means is 5 : 9. Find the value of m.
2. The sum of two numbers is $\dfrac{13}{6}$. An even number of AM's are being inserted between them and their sum exceeds their number by 1. Find the number of means inserted.
3. There are $'n'$ AM's between 3 and 17. If the ratio of the first mean to last mean is 1 : 3, find the value of $'n'$.
4. $'n'$ arithmetic means are inserted in between x and $2y$ and then between $2x$ and y. In case the rth means in each case be equal then find the ratio $\dfrac{x}{y}$.
5. Between two numbers whose sum is $2\dfrac{1}{6}$, an even number of A.M's are inserted. If the sum of these means exceeds their number by unity, then find the numbers means
6. If a, b, c are in A.P and $'P'$ is the A.M between a and b and $'q'$ is the A.M between b and c, then find the A.M between p and q.

C5. Geometric Progression (G.P)

A sequence of terms $t_1, t_2, t_3, ...t_n, ...$ are said to be in G.P if $\dfrac{t_{k+1}}{t_k} =$ constant $\forall\, k \in \mathbb{N}$.

The first term $'t_1'$ is denoted by $'a'$ and the constant ratio between two successive terms by $'r'$ called common ratio.

Then a G.P is of the form $a, ar, ar^2, ...$

Note: All the terms in the G.P are non zero.

nth term of a G.P

The $'n'$th term of a G.P denoted by $'t_n'$ is given by $t_n = a.r^{n-1}$

Sum of 'n' terms of a G.P

The sum of $'n'$ terms of G.P denoted by S_n is given by

$S_n = a + ar + ar^2 + ... + ar^{n-1}$

$$= \begin{cases} na \quad , \quad if \ r = 1 \\ a\left(\frac{1-r^n}{1-r}\right), if \ \ r \neq 1 \end{cases}$$

If $-1 < r < 1$ then r^n tends to zero for infinite value of n

$\therefore$ In this case the sum of infinite terms denoted by $S_\infty = \frac{a}{1-r}, -1 < r < 1$

Practice Problems

1. In a G.P if $t_5 = 81$, $t_2 = 24$ then find S_8

2. In a G.P if $S_3 : S_6 = 125 : 152$ then find 'r'

3. In a G.P, $S_n = 255$, $t_n = 128$, $r = 2$ then find 'n'

4. A G.P consists of an even number of terms. The sum of all the terms is three times that of odd terms. Determine the common ratio of the G.P.

5. In an increasing G.P, the sum of the first and last term is 66, the product of the second and last but one term is 128, and the sum of all the terms is 126. Find the number of terms

6. If S_1, S_2, S_3 be respectively the sums of $n, 2n, 3n$ terms of a G.P then prove that

$S_1^2 + S_2^2 = S_1(S_2 + S_3)$

7. If $S_1, S_2, ..., S_n$ are the sums of infinite geometric series whose first terms are 1,2,3,….,n and common ratio's are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n+1}$ respectively then find

1) $S_1 + S_2 + S_3 + ... + S_n$ 2) $S_1^2 + S_2^2 + S_3^2 + ... + S_n^2$

8. Find $1 + (1+x) + \left(1+x+x^2\right) + \left(1+x+x^2+x^3\right) + ... + n$ terms

9. If $x = 1 + a + a^2 + ... + \infty$

$y = 1 + b + b^2 + ... + \infty$

Where $|a| < 1, |b| < 1$, prove that $1 + ab + a^2b^2 + ... + \infty = \frac{xy}{x+y-1}$

10. Given t_1, t_2, t_3 are in G.P and $t_1, t_2, t_3 - 64$ are in A.P and $t_1, t_2 - 8, t_3 - 64$ are in G.P then find t_1, t_2, t_3

C6. Properties of G.P

1. If $a_1, a_2, a_3, ..., a_{2n}$ are in G.P then

(a) $a_1 a_{2n} = a_2 a_{2n-1} = a_3 a_{2n-2} = ... = a_n a_{n+1}$

(b) $a_n^2 = a_{n-k} a_{n+k} \ \forall \ k = 0$ to $n-1$

2. If three terms a,b,c are in G.P then $b^2 = ac$

3. (a) Three terms in G.P are taken as $\frac{a}{r}, a, ar$

(b) Four terms in G.P are taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

4. If $a_1, a_2, a_3, \ldots\ldots, a_n$ are in G.P then

(a) $a_1k, a_2k, a_3k, \ldots\ldots, a_nk$ are also in G.P $\forall\ k \in \mathbb{R} - \{0\}$

(b) $a_1^k, a_2^k, a_3^k, \ldots\ldots\ldots, a_n^k$ are also in G.P $\forall\ k \in R$

5. If $a_1, a_2, a_3, \ldots\ldots, a_n$ and $b_1, b_2, b_3, \ldots\ldots, b_n$ are in G.P then

$a_1^p b_1^q, a_2^p b_2^q, a_3^p b_3^p, \ldots\ldots\ldots, a_n^p b_n^q$ are also in G.P $\forall p, q \in \mathbb{R}$

Practice Problems

1. If a,b,c,d are in G.P, prove that $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P
2. If three successive terms of a G.P with common ratio $r(r > 1)$ form the sides of a ΔABC then find $[r] + [-r]$ (where $[.]$ denotes the G.I.F)
3. Find the number of increasing G.P(s) with first term unity, such that any three consecutive terms, on doubling the middle become an A.P
4. If a,b,c,d are in G.P, then find $\frac{(a^2+b^2+c^2)(b^2+c^2+d^2)}{(ab+bc+cd)^2}$
5. A G.P consists of even number of terms. If the sum of terms of occupying the odd places is S_1 and that of terms in the even places is S_2. Then find the common ratio of G.P

C7. Geometric Means (GM's)

Let $'a'$ and $'b'$ are two quantities. If $x_1, x_2, \ldots., x_n$ are $'n'$ quantities such that $a, x_1, x_2, x_3, \ldots., x_n, b$ are in G.P then $x_1, x_2, x_3, \ldots\ldots, x_n$ are called $'n'$ G.M's between $'a'$ and $'b'$

Now $b = t_{n+2} = a.r^{n+1}$

$\Rightarrow$ Common ratio $= r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Product of $'n'$ GM's between a & b

$= x_1 x_2 x_3 \ldots.. x_n = (ab)^{\frac{n}{2}}$

Practice Problems

1. If one GM $'G'$ and two AM's $'p'$ and $'q'$ are inserted between two given numbers, then show that $G^2 = (2p - q)(2q - p)$
2. If two geometric means g_1 and g_2 and one arithmetic mean $'A'$ be inserted between two numbers, then find $\frac{g_1^2}{g_2} + \frac{g_2^2}{g_1}$

3. If $\dfrac{a^p+b^p}{a^q+b^q}$ is G.M between a and b and $p-q=1$, then find the value of $'p'$.

4. If a,b,c are three distinct real numbers in G.P such that $a+b+c=2xb$, then find range of $'x'$

5. Let $'S'$ be the sum, $'P'$ be the product and $'R'$ be the sum of reciprocals of $'n'$ terms of G.P the find $'S'$ in terms R and $'P'$

C8. Harmonic Progression (H.P)

A sequence of terms $t_1,t_2,t_3,.....,t_n$ are said to be in H.P if $\dfrac{1}{t_1},\dfrac{1}{t_2},\dfrac{1}{t_3}........,\dfrac{1}{t_n}$ are in A.P

Therefore an Harmonic progression is of the form $\dfrac{1}{a},\dfrac{1}{a+d},\dfrac{1}{a+2d},......$

n^{th} term of H.P

The n^{th} term of an H.P denoted by t_n is given by

$$t_n=\frac{1}{a+(n-1)d}$$

Practice Problems

1. The 8th and 14th term of a H.P are $\dfrac{1}{2}$ and $\dfrac{1}{3}$ respectively. Find its 20th term

2. The mth term of a H.P is $'n'$ and the nth term is $'m'$. Prove that rth term is $\dfrac{mn}{r}$

3. An A.P a G.P and a H.P have a and b for their first two terms. If their $(n+2)^{nd}$ terms are in G.P, then find $\dfrac{b^{2n+2}-a^{2n+2}}{ab\left(b^{2n}-a^{2n}\right)}$

4. If $a_1,a_2,a_3,.......a_n$ are in H.P, then prove that $\dfrac{a_1}{a_2+a_3+.....+a_n}$, $\dfrac{a_2}{a_1+a_3+.....+a_n}$, ……. $\dfrac{a_n}{a_1+a_2+.....+a_{n-1}}$ are in H.P.

5. If positive numbers a,b,c are in H.P, then find the least value of $\dfrac{a+b}{2a-b}+\dfrac{c+b}{2c-b}$

C9. Harmonic Mean (H.M)

If three quantities a,x,b are in H.P then $'x'$ is called H.M of $'a'$ and $'b'$

$\Rightarrow \dfrac{1}{a},\dfrac{1}{x},\dfrac{1}{b}$ are in A.P

$$\Rightarrow \frac{2}{x}=\frac{1}{a}+\frac{1}{b}$$

$$\Rightarrow x=\frac{2ab}{a+b}$$

If $a, x_1, x_2, ..., x_n, b$ are in H.P then $x_1, x_2, ..., x_n$ are called $'n'$ harmonic means (H.M's) between $'a'$ and $'b'$. In this case

$$\frac{1}{b} = \frac{1}{a} + (n+1)d \Rightarrow d = \frac{a-b}{(n+1)ab}$$

Practice Problems

1. If $'H'$ be the H.M between $'a'$ and $'b'$ then prove that $\frac{1}{H-a} + \frac{1}{H-b} = \frac{1}{a} + \frac{1}{b}$

2. The sum of three numbers in H.P is 26 and sum of their reciprocal is $\frac{3}{8}$. Find the numbers

3. If a, b, c are in H.P, prove that $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$

4. The value of $x+y+z$ is 15 if a, x, y, z, b are in A.P, while the value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is $\frac{5}{3}$ if a, x, y, z, b are in H.P. Find a & b

5. If $A_1, A_2; G_1, G_2; H_1, H_2$ be two AM's ; GM's ; HM's respectively of $'a'$ and $'b'$ then prove that $\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$

6. If $a_1, a_2, a_3, ..., a_n$ are in H.P prove that $a_1a_2 + a_2a_3 + ... + a_{n-1}a_n = (n-1)a_1a_n$

7. If a_1, a_2, a_3 are in A.P; a_2, a_3, a_4 are in G.P and a_3, a_4, a_5 are in H.P prove that a_1, a_3, a_5 are in G.P.

8. If a, b, c are in H.P then prove that

(a) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P

(b) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P

(c) $\frac{\sqrt{bc}}{\sqrt{b}+\sqrt{c}}, \frac{\sqrt{ca}}{\sqrt{c}+\sqrt{a}}, \frac{\sqrt{ab}}{\sqrt{a}+\sqrt{b}}$ are in A.P

9. Consider two positive numbers a and b. If A.M of a and b exceeds their G.M by $\frac{3}{2}$ and G.M of a and b exceeds their H.M by $\frac{6}{5}$, then find $a^2 + b^2$

10. Let $'P'$ be the first of $'n'$ A.M's between two numbers a and b and $'q'$ is the 1st of $'n'$ H.M's between the same numbers. Find the internal in $\frac{q}{p}$ lies

C10. Arithmetico-Geometric sequence

A sequence of the form $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, ...$ is called Arthmetico-Geometric sequence.

The nth term of the above sequence $t_n = \{a + (n-1)d\} r^{n-1}$

and the sum of $'n'$ terms S_n is given by

$$S_n = a + (a+d)r + (a+2d)r^2 + ... + (a + (n-1)d) r^{n-1}$$

$$rS_n = ar + (a+d)r^2 + (a+2d)r^3 + ... + (a + (n-1)d) r^n$$

Subtracting

$$S_n - rS_n = a + \left[dr + dr^2 + ... + d_r{}^{n-1}\right] - \left[a + (n-1)d\right]r^n$$

$$(1-r)S_n = a + dr\left(\frac{1-r^{n-1}}{1-r}\right) - \left(a + (n-1)d\right)r^n$$

$$\Rightarrow S_n = \frac{a}{1-r} + dr\frac{1-r^{n-1}}{(1-r)^2} - \frac{\left[a + (n-1)d\right]r^n}{1-r}$$

If $-1 < r < 1$ then the sum of infinite Arithmetico-Geometric series is $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

Practice Problems

1. Find the sum of infinity of the series $1 - 3x + 5x^2 - 7x^3 + ...,|x| < 1$

2. Find the sum to infinity of the series $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + ... + \infty$

3. Find the sum to n terms of the seirs $3 + 5 + 9 + 17 + ...$

4. Find sum to infinity of the series $\frac{2}{3} - \frac{5}{6} + \frac{2}{3} - \frac{11}{24} + ...$

5. Find the sum of first 'n' terms of the series $1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + ...$

C11. $\sum$ Notation

$$\sum_{r=1}^{n} T_r = T_1 + T_2 + ... + T_n$$ where T_n is the general term

Eg1:- $\sum_{r=1}^{4} r^2 = 1^2 + 2^2 + 3^2 + 4^2$

Eg2:- $\sum_{a,b,c} a(b-c) = a(b-c) + b(c-a) + c(a-b)$

Practice Problems

1. Find $\sum_{r=1}^{n} r$ or $\sum n$

2. Find $\sum_{r=1}^{n} r^2$ or $\sum n^2$

3. Find $\sum_{r=1}^{n} r3$ or $\sum n^3$

4. Find $\sum_{r=1}^{n} r(r+1)$

5. Find $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$

ANSWER & KEY

Concept-1

1.	5	2.	0	3.	1	4.	$\frac{4}{5}$
5.	17	6.	9	8.	0		

Concept-2

1.	61	2.	$-(m+n)$	3.	70336	4.	3050
5.	41	6.	25 or 36	7.	$\frac{162}{119}$	9.	$d=2, S_n=n^2$
10.	6	11.	26	12.	80	13.	8
14.	$\frac{mn(mn+1)}{2}$	15.	62525				

Concept-4

1.	14	2.	12	3.	6	4.	$\frac{r}{n+1-r}$

Concept-5

1.	$\frac{3^8-2^8}{8}$	2.	$\frac{3}{5}$	3.	8	4.	2
5.	6	7.	(1) $\sum n$ (2) $\sum n^2-1$				
8.	$\frac{x(x^n-1)-n(x-1)}{(x-1)^2}$			10.	4, 20, 100		

Concept-6

2.	–1	3.	1	4.	1	5.	$\frac{S_2}{S_1}$

Concept-7

2.	$2A$	3.	$\frac{1}{2}$	4.	$\left(-\infty,-\frac{1}{2}\right]\cup\left[\frac{2}{3},\infty\right)$		
5.	$R\times P^{2/n}$						

Concept-8

1.	$\frac{1}{4}$	3.	$\frac{n+1}{n}$	5.	4

Concept-9

2.	6, 8, 12	4.	$a=1, b=9$	9.	159
10.	$\frac{q}{p}$ does not lie between 1 and $\left(\frac{n+1}{n-1}\right)^2$				

Concept-10

1. $\dfrac{1-x}{(1+x)^2}$ 2. $\dfrac{1+x}{(1-x)^3}$ 3. $2^{n+1}+n-2$ 4. $\dfrac{2}{9}$

5. $\dfrac{n(n+1)^2}{2}$ if n is even

   $\dfrac{n^2(n+1)}{2}$ if n is odd

Concept-11

1. $\dfrac{n(n+1)}{2}$ 2. $\dfrac{n(n+1)(2n+1)}{6}$ 3. $\left(\dfrac{n(n+1)}{2}\right)^2$ or $\dfrac{n^2(n+1)^2}{4}$

4. $\dfrac{n(n+1)(n+2)}{3}$ 5. $\dfrac{n(n+3)}{4(n+1)(n+2)}$